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Original Research Article

On a first order linear singular differential equation in the space *K'*

Abdourahman Haman Adji[1,*], **Shankishvili Lamara Dmitrievna**[2]

[1] *Department of Mathematics and Computer Sciences, Faculty of Sciences, University of Ngaoundéré, Ngaoundéré 454, Cameroon*

[2] *Department of Mathematics, Georgian Technical University, Tbilisi 0171, Georgia*

*** Corresponding author:** Abdourahman Haman Adji, abdoulshehou@yahoo.fr

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ABSTRACT: We propose in this work to describe all the generalized-function solutions of the non-homogeneous first-order linear singular differential equation with A, B two real numbers, s and $p \in \mathbb{N}, n \geq 1, q \in Z_+$, in the space of generalized functions *K'*. In the case of a second right-hand side consisting of an *s-order* derivative of the Dirac-delta function, we have completely investigated the considered equation when we look for the solution in the form of $y(x) = \sum_{k=0}^{N} C_k\, \delta^{(k)}(x)$ with the unknown coefficients C_k which we have determined case by case, taking into account the relationship between the parameters inside. On the basis of what has been done, we focus our present research on applying the principle of superposition of the solutions that is conducting us to the awaited result when we also maintain the classical solutions of the homogeneous equation which remains the same.

KEYWORDS: test functions; generalized functions; Dirac-delta function; Fourier transform; zero-centered solution

MSC CLASSIFICATION: 34A05; 39A05; 42B10; 44A05.

1. Introduction

Solving differential equations is already a great step toward understanding some physical phenomena described by the constructed models of real life. For that aim, we use different theories and methods when we look for solutions to differential equations, and this is also well known and illustrated in some books reflected in scientific research done by scientists. Among many others, we can cite the method of substituting the variable, numerical methods, and analytical methods when solving some kind of differential equation.

We can stipulate in a very simple way that the principle of superposition of solutions of a differential equation is the sum of two or more solutions of the considered equation, which should be exactly, and once again, a solution. We recall that, from the general theory of differential equations, the expression of the following form $\sum_{i=0}^{n} c_i b_i(t) y^{(i)}(t) = f(t)$, where the variables coefficients $c_i b_i(t),\ i = 0, 1, \ldots, n,\ f(t)$ are continuous real functions with $b_n(t) \neq 0, c_i$ arbitrary constants is called an n order linear differential equation. When we set up a problem as follows: solve a differential equation of the form $\sum_{i=0}^{n} c_i b_i(t) y^{(i)}(t) = \sum_{j=0}^{m} f_j(t)$ where $f_j(t)$ (for every j= $0, \ldots, m$) are well-known functions then it is clear that in practice one should solve one by one the differential equation $\sum_{i=0}^{n} c_i b_i(t) y^{(i)}(t) = f_j(t)$ (for every j= $0, \ldots, m$). Supposing that each partial equation has a solution defined by $y_j(t)$ (for every j= $0, \ldots, m$) then, we know that the general solution of the investigated initial equation should be $\sum_{j=0}^{m} y_j(t)$.

However, let us mention that sometimes solving even a first-order linear differential equation in some specific cases may not be easy. Many scientific papers have recently been devoted to considerable interest in problems concerning the existence of solutions to differential and functional differential equations (FDE) in various known spaces of generalized functions.

As we notice, a lot of serious areas in theoretical and mathematical physics, the theory of partial differential equations, quantum electrodynamics, operational calculus, and functional analysis widely use the methods of the distributions theory.

One can clearly understand and underline the importance of the very rich scientific theory of generalized functions nowadays used in the development of several areas of mathematics and also globally in sciences in general. For example, we note that the Dirac delta function plays a fundamental role in all this theory, and, among many others, the works of Mikio Sato have glimpsed new paths in algebraic analysis in the so-called theory of hyperfunctions widely investigated.

Recent scientific research on generalized functions in noncommutative algebras, where the Dirac operator is the evocative idea of this new formulation to arrive at the solution of the problem, has been studied by D. Alfonso Santiesteban et al.[1–5].

This work is devoted to the question of the investigation of the solvency in the space of generalized functions *K'*, a linear differential equation of the first-order with singularity and a finite linear combination of Dirac-delta functions and its derivatives on the second-hand side. Namely, we consider the equation of the following type.

$$Ax^p y'(x) + Bx^q y(x) = \sum_{s=0}^{n} \delta^{(s)}(x) \tag{1}$$

where A, B are real numbers and, $p \in \mathbb{N}, n \geq 1, q, s \in \mathbb{N} \cup \{0\}$.

As we mentioned in our previous research among others, we can note similar investigations recently done by various authors such as Liangprom and Nonlaopon[6], Jhanthanam et al.[7] and we completely investigated the same question when we had only one term in the right-hand side denoted $\delta^{(s)}(x)$. For full details, one can refer to article[8]. The same ideas when seeking the general solutions of the considered equation remain as done, using now more precisely in this work the principle of superposition of the solutions to reach the goal.

From the resolution of the simple differential equation

$$x^n y(x) = \delta^{(s)}(x) \tag{2}$$

whose distributional solution function is defined by the expression

$$y(x) = \frac{\delta^{(s)}(x)}{x^n} = \frac{(-1)^n s!\, \delta^{(n+s)}(x)}{(s+n)!} + \sum_{k=1}^{n} c_k \delta^{(k-1)}(x) \tag{3}$$

where c_k are arbitrary constants, we understand the issue and the challenge that looms on the horizon with regard to the resolution of an equation of general order m, i.e., of the type $x^n y^{(m)}(x) = \delta^{(s)}(x)$, whose distributional solution will naturally be obtained and defined by a rather enormous formula just in the case mentioned. This is therefore a reason for seeking to generalize the resolution of certain types of equations thus evoked and related to our work. The technicality and mastery of the techniques of derivations and integration in the sense of the distributions are taken into account in the realization of this work.

Therefore, within this research, we give a complete description of the solutions of Equation (1) in the space *K'* applying the principle of superposition of solutions (in the case of solvency).

We organize the paper as follows: first of all, in Section 2, we present the necessary concepts and notions from the theory of linear differential equations and also well-known distributional theory. Sections 3 and 4 devoted to the main results obtained in this work begin separately with the degenerate case when $AB = 0$ where we describe the solutions of Equation (1), centered at zero. Next, it is constructed the general solution as the union of generalized and classical solutions. In Section 5 dedicated to the non-degenerate case $AB \neq 0$, we describe the classical solutions of Equation (1) in $\mathbb{R}^*$. Section 6 titled Conclusion, which concludes this paper.

2. Preliminaries

For the execution of this research, it is needed some important reminders related to the notions of the Fourier transform, its properties, and generalized function centered at a given point. For full details on these facts, we refer to[9–12] all that K is denoted the space of test functions, of indefinitely differentiable and identically zero outside a bounded set on $\mathbb{R}^1$ functions and K' the space of generalized functions on K. For function $\varphi(t) \in K$, through $F\varphi = \hat{\varphi}$, we denoted the Fourier transform defined by the equation

$$(F\varphi)(x) = \hat{\varphi}(x) = \int_{-\infty}^{+\infty} \varphi(t)e^{ixt}dt \tag{4}$$

The Fourier transform of the generalized function $f \in K'$ we define by the rule (Parseval equality):

$$\left(\hat{f}, \hat{\varphi}\right) = 2\pi(f, \varphi) \tag{5}$$

For the Fourier transform of generalized function, many properties are conserved as those taking place for Fourier transform for test functions, and particularly equations relationship between differentiability and decrease meant. From them in particular it follows that:

$$F\left[\delta^{(s)}(t)\right] = (-ix)^s, s \in \mathbb{N}\cup\{0\} \tag{6}$$

We need the following assertions, which can be found along with their proofs in books on the theory of generalized functions. For example, refer to[11,13–15].

Theorem 1. *If $f, g \in K'$ and $f' = g'$ then $f - g = c$.*

Theorem 2. *Let $A(x) \in C^\infty(\mathbb{R}^1)$. The differential equations $y' = A(x)y$ in the space K' does not admit other solutions which are not classical solutions.*

Definition 1. *Generalized function $f \in K'$ is called centered at the point x_0, if $\left(f, \varphi(x)\right) = 0$ for all $\varphi(x) \in K$ such $x_0 \in Supp\varphi$.*

Theorem 3. *Let $f \in K'$ centered at zero. Then there exist $m \in \mathbb{N} \cup \{0\}$ such that:*

$$f(x) = \sum_{j=0}^{m} c_j \delta^{(j)}(x) \tag{7}$$

where c_j are some constants.

Lemma 1. *Let $\beta(x) \in C^\infty(\mathbb{R}^1)$. Then it holds the equation*

$$\beta(x)\delta^{(j)}(x)=\sum_{j=0}^{l}(-1)^{j}C_l^j\beta^{i}(0)\,\delta^{(l-j)}(x). \tag{8}$$

As for the proof of the Lemma 1 as well as all the details relating to it, we can refer to[16]

As a consequence from Lemma 1 when $\beta(x)=x^k$, we obtain the following assertion.

Lemma 2. *Let* $s\in\mathbb{N}\cup\{0\}$, *Then*

$$x^k\delta^{(s)}(x)=\begin{cases}0, & if\ s<k\\ \dfrac{(-1)^k s!}{(s-k)!}\delta^{(s-k)}(x), & if\ s\geq k.\end{cases} \tag{9}$$

Sometimes, we need in our investigation the following expression $\frac{g^{(s)}(x)}{x^n}$. We will understand this expression in the sense of the following definition.

Definition 2. *The quotient* $\frac{\delta^{(s)}(x)}{x^n}$ *is called a generalized function* $y(x)\in K'$ *which satisfy in the space* K' *the equality* $x^n y(x)=\delta^{(s)}(x)$ when it is realized the following:

$$\left(x^n y(x),\varphi(x)\right)=\left(\delta^{(s)}(x),\varphi(x)\right),\varphi\in K. \tag{10}$$

Let us from Lemma 2 and Definition 2 deduce the following equation for the computation of the generalized function defined by the expression: $\frac{\delta^{(s)}(x)}{x^n}$.

Lemma 3. *Let* $n\in\mathbb{N}, s\in\mathbb{N}\cup\{0\}$. *It holds the following equation.*

$$\frac{\delta^{(s)}(x)}{x^n}=\frac{(-1)^n s!\,\delta^{(n+s)}(x)}{(s+n)!}+\sum_{k=1}^{n}c_k\delta^{(k-1)}(x) \tag{11}$$

Proof. Let us use the Definition 2 and apply the Fourier transform to both sides of the equality $x^n y(x)=\delta^{(s)}(x)$ with consideration that $\widehat{(\imath x)^n y}(x)=\widehat{y(x)}(\varepsilon)$. Finally we have $\hat{y}^n(\varepsilon)=\widehat{(\imath x)^n y}(x)=i^n\widehat{\delta^{(s)}}(x)=i^n(-i\varepsilon)^s$. From the previous we reach $\hat{y}(\varepsilon)=P_{n-1}(\varepsilon)+\frac{(-1)^n s!}{(s+n)!}(-i\varepsilon)^{s+n}$ where $P_{n-1}(\varepsilon)$ is a polynomial with arbitrary coefficients. Applying the inverse Fourier transform with respect to the Equation (6) we reach the needed result and arrive to the Equation (11). So that consequently the lemma is proved.

Remark 1. *We underline that in the result of the «quotient» of the generalized function* $\delta^{(s)}(x)$ *by* x^n *because of the "union" of singularity, it is not only increasing the order of the derivative of delta function, but it arises some arbitrary depending of* n.

3. The degenerate case

First of all let us consider at the beginning of this investigation Equation (1) in the case we called degenerate case, when it is realized the condition:

$$AB=0 \tag{12}$$

This situation is almost simple and leads us to obtain completed results as consequence of Lemma 3.

Theorem 4. *Let* $q,s\in\mathbb{N}\cup\{0\}, n\geq 1$. *The general solution of equation*

$$Bx^q y(x)=\sum_{s=0}^{n}\delta^{(s)}(x) \tag{13}$$

has the following form

$$y(x) = \sum_{s=0}^{n} \frac{(-1)^q s!}{B(s+q)!} \delta^{(s+q)}(x) + \sum_{k=1}^{q} c_k \delta^{(k-1)}(x) \tag{14}$$

where c_k $(k = 1, \ldots, q)$ *are arbitrary constants. Let state a clear formalism useful here and further. We set*

$$\sum_{k=m}^{n} \delta^{(k)}(x) = 0, \qquad \text{if } n < m \tag{15}$$

In another way, we can see that the Theorem 4 is just a reformulation of the Lemma 3.

Next, we move to the following theorem.

Theorem 5. *Let* $p \in \mathbb{N},\ s \in \mathbb{N} \cup \{0\},\ n \geq 1$. *The general solution of the equation*

$$Ax^p y'(x) = \sum_{s=0}^{n} \delta^{(s)}(x) \tag{16}$$

has the following form defined by:

$$y(x) = \sum_{s=0}^{n} \frac{(-1)^p s!}{A(s+p)!} \delta^{(s+p-1)}(x) + \sum_{k=1}^{p-1} c_{k+1} \delta^{(k-1)}(x) + c_1 \theta(x) + c_0 \tag{17}$$

where c_k $(k = 0, \ldots, p)$ *are arbitrary constants and* $\theta(x)$ *is the Heaviside test function.*

The proof of this theorem also can be deduced from the Equation (11) and for more details refer to[8] when we had only $\delta^{(s)}(x)$ as second member of the investigated equation.

In fact, in that case, we have investigated the Equation (16) with complete details and for full comprehension refer to[8]

$$y' = \frac{1}{A} \frac{\delta^{(s)}(x)}{x^p} = \frac{(-1)^p s!}{A(s+p)!} \delta^{(s+p)}(x) + \sum_{k=1}^{p} c_k \delta^{(k-1)}(x)$$

from there by integration, with respect to $\delta = \theta'$ we arrive to Equation (17) with respect to the principle of superposition of solutions. The theorem is proved.

Let us move to the next interesting section.

4. The non-degenerate case

In this part, we undertake the most important and very interesting study of the considered equation, which is related to the situation when it is realized the following condition

$$AB \neq 0 \tag{18}$$

Let's call this case a non-degenerate case.

First of all, we are going to formalize the following important theorem which gives the necessary conditions for the solvency of the Equation (1) in K'.

Theorem 6. *Let* $\neq 0,\ p \in \mathbb{N}, n \geq 1, q \in \mathbb{N} \cup \{0\}$. *For the solvency of the Equation (1) in the space* K', *it is necessary and sufficient that*

$$(q-p+1)^2 + (B - A(q-p+1)^2) \neq 0 \tag{19}$$

Proof. For this proof, refer to[8], knowing that the particular solution $y(x)$ of the non-homogeneous Equation (1) should be a functional centered in zero of the form of such following defined functional

$$y(x) = \sum_{j=0}^{N} c_j \delta^{(j)}(x) \tag{20}$$

where N is a sufficiently great number and suppose contrary Equation (19), i.e., when

$$q = p - 1, \qquad B = A(q + s + 1), \tag{21}$$

after substitution Equation (21) into the equation and with the application of Lemma 2 we reach the following result:

$$Ax^p \sum_{j=0}^{N} c_j \delta^{(j+1)}(x) + Bx^q \sum_{j=0}^{N} c_j \delta^{(j)}(x) = \sum_{s=0}^{n} \delta^{(s)}(x)$$

From the previous, we obtain:

$$A \sum_{j=p-1}^{N} \frac{(-1)^p (j+1)!}{(j+1-p)!} c_j \delta^{(j+1-p)}(x) + B \sum_{j=q}^{N} \frac{(-1)^q j!!}{(j-q)!} c_j \delta^{(j+q)}(x) = \sum_{s=0}^{n} \delta^{(s)}(x) \tag{22}$$

or after arrangement

$$A \sum_{j=0}^{N-(p-1)} c_{j+p-1} \frac{(-1)^p (j+p)!}{j!} \delta^{(j)}(x) + B \sum_{j=0}^{N-q} \frac{(-1)^q (j+q)!}{j!} c_{j+q} \delta^{(j)}(x) = \sum_{s=0}^{n} \delta^{(s)}(x) \tag{23}$$

From our previous results published in the paper[8] we got the conditions of solvency of the non-homogeneous algebraic system, which appears from Equation (23) in this case, when we had only the right-hand side of Equation (1) only $\delta^{(s)}(x)$. All that has been said and done leads us, by the application of the principle of superposition of solutions, to the following important results.

These results are formalized in situations where the parameters *p, q, A, B*, and *s* are connected by some specific conditions, deduced systematically when analyzing the system arising in the process of the research of the needed solutions.

Theorem 7. *Let* $AB \neq 0$; $p \in \mathbb{N},\ q,\ s \in \mathbb{N} \cup \{0\}$ *and be realized the condition* $q = p - 1, B \neq A(q + s + 1)$. *Then, the centered in zero general solution of the Equation (1) has the following form:*

$$y(x) = \sum_{s=0}^{n} \frac{(-1)^q s!}{(s+q)!\,[B - A(s+q+1)]} \delta^{(q+s)}(x) + \sum_{j=0}^{q-1} c_j \delta^{(j)}(x) \tag{24}$$

where c_j $(j = 0, \dots, q-1)$ *are arbitrary constants in the case when*

$$B - A(j + q + 1) \neq 0,\ j \in \mathbb{Z}_+. \tag{25}$$

And if there exists $j_* \in \mathbb{Z}_+ \backslash \{s\}$, *such that*

$$B - A(j_* + q + 1) \neq 0, \qquad j_* \in \mathbb{Z}_+ \backslash \{s\} \tag{26}$$

then, the solution has the following form

$$y(x) = \sum_{s=0}^{n} \frac{(-1)^q s!}{(s+q)!\,[B - A(s+q+1)]} \delta^{(q+s)}(x) + \sum_{j=0}^{q-1} c_j \delta^{(j)}(x) + c_{j_*+q} \delta^{(j_*+q)}(x) \tag{27}$$

where c_j $(j = 0, \dots, q-1)$, c_{j_*+q} *are arbitrary constants.*

Proof. Applying the principle of superposition of solutions on the basis of our previous results we refer to the paper[8] we arrived easily at the needed result.

Analogically as in the simple case, let us now move to the investigation of the most difficult and interesting case when it is realized the condition: $q \neq p - 1$.

For this aim let's formalize the following:

Theorem 8. *Let be fulfilled the condition* $AB \neq 0, p \in \mathbb{N}, n \geq 1,\, q, s \in \mathbb{N} \cup \{0\}$ *and realized the inequality*

$$q < p - 1. \tag{28}$$

Then, the centered in zero general solution of the Equation (1) is defined by the next formula:

$$y(x) = \sum_{s=0}^{n} \frac{(-1)^q s!}{B(s+q)!} \delta^{(q+s)}(x) + \sum_{j=0}^{q-1} c_j \delta^{(j)}(x) + \\ \sum_{s=0}^{n} \frac{(-1)^q s!}{B(s+q)!} \sum_{l=1}^{[\frac{s}{p-1-q}]} (-1)^{(p-1-q)^l} (\frac{A}{B})^l \, \delta^{(q+s-l(p-1-q))}(x) \times \prod_{m=l}^{l} \frac{[s+p-m(p-1-q)]!}{[s+q-m(p-1-q)]!} \tag{29}$$

where c_j $(j = 0, \ldots, q-1)$ *are arbitrary constants.*

Proof. It is the same as in the previous theorem as we particular have done in the paper[8] remembering that, the solution used was in the following form

$$y(x) = \sum_{l=0}^{[\frac{s}{p-1-q}]} \gamma_l \, \delta^{(q+s-l(p-1-q))}(x) \tag{30}$$

with unknown coefficients γ_l.

Analogically, it can be investigated the case in the contrary inequality. All what has been done conduct us to these following results devoted to the next theorems. Namely, it takes place. □

Theorem 9. *Let* $AB \neq 0;\, p \in \mathbb{N},\; n \geq 1;\, q, s \in \mathbb{N} \cup \{0\}$ *and realized the condition.*

$$q > p - 1. \tag{31}$$

Then, the centered in zero general solution of Equation (1) is given by the following formula:

$$y(x) = \sum_{j=0}^{p-2} c_j \delta^{(j)}(x) + \sum_{s=0}^{n} \frac{(-1)^p s!}{A(s+p)!} \delta^{(s+p-1)}(x) + \\ \sum_{s=0}^{\mathrm{n}} \frac{(-1)^p s!}{A(s+p)!} \sum_{l=1}^{[\frac{s}{q-p+1}]} (\frac{A}{B})^l (-1)^{l(q-p+1)} \, \delta^{(s+p-1-l(q-p-1))}(x) \times \prod_{m=1}^{l} \frac{[s+q-m(q-p+1)]!}{[s+p-m(q-p+1)]!}, \tag{32}$$

where c_j $(j = 0, \ldots, p-2)$ *— are arbitrary constants.*

Proof. Let us conduct this proof when we consider having only an *s*-order of the delta-Dirac function i.e., $\delta^{(s)}(x)$ in the right-hand side of the investigated equation and from that, it consequently be deduced obtained the global result by applying the principle of superposition.

In fact, it is sufficient to find the particular solution of Equation (1) in this case evocated and for this aim, one can refer to[8] and set the form of the distributional function as follows:

$$y(x) = \sum_{l=0}^{[\frac{s}{q-p+1}]} \gamma_l \, \delta^{(s+p-1-l(q-p-1))}(x)$$

with unknown coefficient γ_l. Putting it into Equation (1) with only an s-order of the delta-Dirac function i.e., $\delta^{(s)}(x)$ in the right hand side we obtain

$$Ax^p \sum_{l=0}^{[\frac{s}{q-p+1}]} \gamma_l \delta^{(s+p-l(q-p+1))}(x) + Bx^q \sum_{l=0}^{[\frac{s}{q-p+1}]} \gamma_l \delta^{(s+q-1-(l+1)(q-p+1))}(x) = \delta^{(s)}(x) \tag{a}$$

From the previous, we now reach the following recurrent relationships:

$$\begin{cases} \gamma_0 = \dfrac{(-1)^p s!}{B(s+q)!}; \\ \gamma_l = \dfrac{(-1)^{q-p+1}B[s+q-l(q-p+1)]!\,\gamma_{l-1}}{A[s+p-l(q-p+1)]!}, \qquad l = 1,2,\ldots,\left[\dfrac{s}{p-1-q}\right] \end{cases} \tag{b}$$

This leads us to the final result:

$$\gamma_l = \gamma_0\left(\frac{(-1)^{q-p+1}B}{A}\right)^l \prod_{m=1}^{l} \frac{[s+q-m(q-p+1)]!}{[s+p-m(q-p+1)]!}, \qquad l = 1,\ldots,\left[\frac{s}{q-p+1}\right]. \tag{c}$$

The theorem is proved.

Now let us move to the following important section.

5. Classical solutions of the homogeneous Equation (1) and final results

The next short section presents important results on the basis of already found classical solutions taken into account.

As obtained in our previous research, all the classical solutions of the homogeneous equation in the case $AB \neq 0$ remain the same. For full details refer to[8]

Let's formalize the following important definition related to Equation (1).

Definition 3. Equation (1) is called particular when the conditions p = q +1 and B = A(q + s + 1) are realized and in the contrary case, it is called nonparticular. Then, let us formalize in the following global Theorem 10, the final results of our investigation by combining all that has been obtained in the three previous theorems.

Theorem 10. *The general solution of the non-particular Equation (1) when* $A.B \neq 0,\ p \in \mathbb{N},\ n \geq 1,\ q,s \in \mathbb{N} \cup \{0\}$ *has the following forms:*

1) *If* $q > p-1$

$$y(x) = k_1 e^{-\frac{Bx^{q-p+1}}{A(q-p+1)}}\theta(x) + k_2 e^{-\frac{Bx^{q-p+1}}{A(q-p+1)}}\theta(-x) + \sum_{j=0}^{p-2} c_j \delta^{(j)}(x) + \sum_{s=0}^{n} \frac{(-1)^p s!}{A(s+p)!}\delta^{(s+p-1)}(x) + \sum_{s=0}^{n} \frac{(-1)^p s!}{A(s+p)!}\sum_{l=1}^{\left[\frac{s}{q-p+1}\right]}\left(\frac{B}{A}\right)^l(-1)^{l(q-p+1)} \times \prod_{m=1}^{l}\frac{[s+q-m(q-p+1)]!}{[s+p-m(q-p+1)]!}\delta^{(s+p-1-l(q-p-1))}(x), \tag{33}$$

where $k_1, k_2, A, B, c_0, \ldots, c_{p-2}$ *are arbitrary constants.*

2) *If* $q < p-1,\ \frac{B}{A} < 0,$ *and* $q-p$ *odd number, then we have:*

$$y(x) = k_1 e^{-\frac{Bx^{q-p+1}}{A(q-p+1)}}\theta(x) + k_2 e^{-\frac{Bx^{q-p+1}}{A(q-p+1)}}\theta(-x) + \sum_{j=0}^{q-1} c_j \delta^{(j)}(x) + \sum_{s=0}^{n} \frac{(-1)^q s!}{A(s+q)!}\delta^{(s+q)}(x)\frac{(-1)^q s!}{A(s+q)!}\delta^{(s+q)}(x) + \sum_{s=0}^{n} \frac{(-1)^q s!}{A(s+q)!}\sum_{l=1}^{\left[\frac{s}{p-1-q}\right]}\left(\frac{A}{B}\right)^l(-1)^{l(p-1-q)} \times \prod_{m=1}^{l}\frac{[s+p-m(p-1-q)]!}{[s+q-m(p-1-q)]!}\delta^{(s+q-l(p-1-q))}(x) \tag{34}$$

where $k_1, k_2, A, B, c_0, \ldots, c_{q-1}$ *are arbitrary constants.*

3) *If* $q < p-1,\ \frac{B}{A} > 0$ *and* $q-p$ *even number, then we obtain:*

$$y(x) = k_1 e^{-\frac{Bx^{q-p+1}}{A(q-p+1)}}\theta(x) + \sum_{s=0}^{n} \frac{(-1)^q s!}{A(s+q)!}\delta^{(s+q)}(x) + \sum_{j=0}^{q-1} c_j\delta^{(j)}(x) + \sum_{s=0}^{n}\frac{(-1)^q s!}{B(s+q)!}\sum_{l=1}^{[\frac{s}{p-1-q}]}(\frac{A}{B})^l(-1)^{l(p-1-q)} \times \prod_{m=1}^{l}\frac{[s+p-m(p-1-q)]!}{[s+q-m(p-1-q)]!}\delta^{(s+q-l(p-1-q))}(x) \tag{35}$$

where $k_1, k_2, A, B, c_0, \dots, c_{q-1}$ are arbitrary constants.

4) *If $q < p-1, \frac{B}{A} > 0$ and $q-p$ odd number, then it follows:*

$$y(x) = \sum_{s=0}^{n}\frac{(-1)^q s!}{B(s+q)!}\delta^{(s+q)}(x) + \sum_{j=0}^{q-1} c_j\delta^{(j)}(x) + \sum_{s=0}^{n}\frac{(-1)^q s!}{B(s+q)!}\sum_{l=1}^{[\frac{s}{p-1-q}]}(\frac{B}{B})^l(-1)^{l(p-1-q)} \times \prod_{m=1}^{l}\frac{[s+p-m(p-1-q)]!}{[s+q-m(p-1-q)]!}\delta^{(s+q-l(p-1-q))}(x), \tag{36}$$

where $k_1, k_2, A, B, c_0, \dots, c_{q-1}$ are arbitrary constants.

5) *If $q < p-1, \frac{B}{A} > 0$ and $q-p$ even number, then we have:*

$$y(x) = k_2 e^{-\frac{Bx^{q-p+1}}{A(q-p+1)}}\theta(-x) + \sum_{s=0}^{n}\frac{(-1)^q s!}{B(s+q)!}\delta^{(s+q)}(x) + \sum_{j=0}^{q-1} c_j\delta^{(j)}(x) + \sum_{s=0}^{n}\frac{(-1)^q s!}{B(s+q)!}\sum_{l=1}^{[\frac{s}{p-1-q}]}(\frac{A}{B})^l(-1)^{l(p-1-q)} \times \prod_{m=1}^{l}\frac{[s+p-m(p-1-q)]!}{[s+q-m(p-1-q)]!}\delta^{(s+q-l(p-1-q))}(x) \tag{37}$$

where $k_2, A, B, c_0, \dots, c_{q-1}$; are arbitrary constants.

Finally, we now move to the next following situation.

6) The situation when $q = p-1,\ B \neq A(q+s+1)$

(1) If $B - A(j+q+1) \neq 0\ \forall j \in \mathbb{Z}_+$ and $\frac{B}{A} < 1$,

$$y(x) = \sum_{s=0}^{n}\frac{(-1)^q s!}{(s+q)!\,[B-A(q+s+1)]}\delta^{(s+q)}(x) + \sum_{j=0}^{q-1} c_j\,\delta^{(j)}(x) + k_1 x^{-\frac{B}{A}}\theta(x) + k_2|x|^{-\frac{B}{A}}\theta(-x) \tag{38}$$

where $k_1, k_2, A, B, c_0, \dots, c_{q-1}$; are arbitrary constants.

(2) If $\exists\ j_* \in \mathbb{Z}_+\backslash\{s\}$ such that, $B - A(j_*+q+1) = 0$ and $\frac{B}{A} < 1$,

$$y(x) = \sum_{s=0}^{n}\frac{(-1)^q s!}{(s+q)![B-A(q+s+1)]}\delta^{(s+q)}(x) + c_{j_*+q}\delta^{(j_*+q)}(x) + \sum_{j=0}^{q-1} c_j\,\delta^{(j)}(x) + k_1 x^{-\frac{B}{A}}\theta(x) + k_2|x|^{-\frac{B}{A}}\theta(-x), \tag{39}$$

where $k_1, k_2, A, B, c_0, \dots, c_{q-1}, c_{j_*+q}$; are arbitrary constants.

(3) If $B - A(j+q+1) \neq 0,\ \forall\ j \in \mathbb{Z}_+$ and $\frac{B}{A} \geq 1$,

$$(x) = \sum_{s=0}^{n}\frac{(-1)^q s!}{(s+q)!\,[B-A(q+s+1)]}\delta^{(s+q)}(x) + \sum_{j=0}^{q-1} c_j\,\delta^{(j)}(x), \tag{40}$$

where $A, B, c_0, \dots, c_{q-1}$; are arbitrary constants.

(4) If $\exists\ j_* \in \mathbb{Z}_+\backslash\{s\}$ such that, $B - A(j_*+q+1) = 0$ and $\frac{B}{A} \geq 1$,

$$y(x) = \sum_{s=0}^{n}\frac{(-1)^q s!}{(s+q)!\,[B-A(q+s+1)]}\delta^{(s+q)}(x) + c_{j_*+q}\delta^{(j_*+q)}(x) + \sum_{j=0}^{q-1} c_j\,\delta^{(j)}(x) \tag{41}$$

where $c_{j_*+q}, A, B, c_0, \dots, c_{q-1}$; are arbitrary constants.

Let us do this important remark before concluding our work.

Remark 2. *As we note, the expressions defining the general solutions of the investigated equation are quite huge and one should verify that these obtained formulas do satisfy the identity left part equal to the right one.*

For example, in the case when $q < p - 1$, $\frac{B}{A} > 0$ and $q - p$ even number, we should replace the Equation (37) defined upstairs of the general solution in the Equation (1), and we must obtain exactly after all calculations

$$Ax^p y'(x) + Bx^q y(x) = \sum_{s=0}^{n} \delta^{(s)}(x).$$

This means, If $q < p - 1$, $\frac{B}{A} > 0$ and $q - p$ even number then, the solution defined by the Equation (37) verifies Equation (1), i.e., $y(x) \in K'$ and $\forall \varphi(x) \in K$ we should have:

$$(Ax^p y'(x) + Bx^q y(x), \varphi(x)) = (\sum_{s=0}^{n} \delta^{(s)}(x), \varphi(x)) = \sum_{s=0}^{n} (-1)^{(s)} \varphi^{(s)}(0). \tag{42}$$

By virtue of the linearity we reach the following step:

$$\left(Ax^p y'(x), \varphi(x)\right) + \left(Bx^q y(x), \varphi(x)\right) = \sum_{s=0}^{n} (-1)^{(s)} \varphi^{(s)}(0). \tag{43}$$

Otherwise using the property of the derivative of a distribution we have:

$$\left(y'(x), Ax^p \varphi(x)\right) + \left(y(x), Bx^q \varphi(x)\right) = \sum_{s=0}^{n} (-1)^{(s)} \varphi^{(s)}(0). \tag{44}$$

$$-\left(y(x), \left(Ax^p \varphi(x)\right)'\right) + \left(y(x), Bx^q \varphi(x)\right) = \sum_{s=0}^{n} (-1)^{(s)} \varphi^{(s)}(0). \tag{45}$$

the previous leads us to the next result:

$$\left(y(x), -(Ax^p \varphi'(x) + Apx^{p-1} \varphi(x)\right) + Bx^q \varphi(x) = \sum_{s=0}^{n} (-1)^{(s)} \varphi^{(s)}(0). \tag{46}$$

After little arrangement we obtain:

$$\left(y(x), -Ax^p \varphi'(x) + (Bx^q - Apx^{p-1}) \varphi(x)\right) = \sum_{s=0}^{n} (-1)^{(s)} \varphi^{(s)}(0). \tag{47}$$

Finally, substituting the solution by its analytical formula we should get:

$$\Bigg(k_1 e^{-\frac{Bx^{q-p+1}}{A(q-p+1)}} \theta(x) + \sum_{s=0}^{n} \frac{(-1)^q s!}{A(s+q)!} \delta^{(s+q)}(x) + \sum_{j=0}^{q-1} c_j \delta^{(j)}(x) + \sum_{s=0}^{n} \frac{(-1)^q s!}{B(s+q)!} \sum_{l=1}^{\left[\frac{s}{p-1-q}\right]} \left(\frac{A}{B}\right)^l (-1)^{l(p-1-q)} \times \prod_{m=1}^{l} \frac{[s+p-m(p-1-q)]!}{[s+q-m(p-1-q)]!} \delta^{(s+q-l(p-1-q))}(x), -Ax^p \varphi'(x) + (Bx^q - Apx^{p-1}) \varphi(x) \Bigg) = \sum_{s=0}^{n} (-1)^{(s)} \varphi^{(s)}(0). \tag{48}$$

where $k_1, k_2, A, B, c_0, \ldots, c_{q-1}$ are arbitrary constants.

6. Conclusion

Completely achieved in this paper is a whole description of all the generalized-function solutions of the linear singular differential equation of the first order in the space of generalized functions *K'* with a second right-hand side in the form of a finite linear combination of the Dirac-delta functions and their derivatives. Based on our previous research, we applied case by case, the well-known principle of superposition of the solutions of a differential equation, taking into account the various relationships between the parameters of the studied equation. All the results obtained are described in the theorems we formulated within the paper. At the end of the work we write the lines of the verification of the general solution of the investigated equation in one describe the case.

Author contributions

Conceptualization, AHA; methodology, AHA; software and validation, AHA and SLD; formal analysis and investigation, AHA; resources, AHA; writing—original draft preparation, AHA; writing—review and editing, AHA and SLD; supervision, AHA; funding acquisition, AHA. All authors have read and agreed to the published version of the manuscript.

Conflict of interest

The authors declare no conflict of interest.

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